Stability Analysis of the MUSCL Method on General Unstructured Grids for Applications to Compressible Fluid Flow

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Groupe de travail << Méthodes Numériques >>
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Outline

1. Context
   - The Software Package CEDRE
   - Spatial Discretization on General Polyhedral Meshes

2. Stability of Unstructured MUSCL Schemes
   - Instabilities in Three-dimensional Applications
   - Preliminary Investigation
   - Reconstruction and Stability on Unstructured Grids
   - Applications to Gas Dynamics
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The Software Package CEDRE
Overview of CEDRE

Overview of CEDRE

- Software for numerical simulation of internal flows.
- Solver for compressible Navier-Stokes (RANS and LES).
- Multi-physics: gas dynamics, heat conduction, particles, radiation, reactive flows.
- Used by industry (EADS, SNECMA, SNPE) and research organizations.

Important Feature

- Discretization on general unstructured meshes: Important for industrial applications.
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Spatial Discretization
Basic Principle of the MUSCL Finite-Volume Scheme

1. Reconstruct a piecewise linear solution from the cell averages.
2. Use it to compute upwind fluxes at the cell interfaces.
3. The fluxes determine the dynamics of the cell averages.
Geometry of Unstructured Meshes

- cell $\mathcal{T}_\alpha$ and face $A_{\alpha\beta}$
- for convenience
  $n_{\alpha\beta} = k_{\alpha\beta} = 0$ if $\mathcal{T}_\alpha$ not a neighbor of $\mathcal{T}_\beta$
- fundamental relation
  $\sum_\beta n_{\alpha\beta} = 0$
Reconstruct a gradient $\sigma_\alpha$ in each cell...

$$\sigma_\alpha = \sum_{\beta} s_{\alpha\beta} (u_\beta - u_\alpha)$$

...to compute second order accurate values at the cell interfaces

$$u_{\alpha\gamma} = u_\alpha + k_{\alpha\gamma} \cdot \sigma_\alpha$$
Consistent Reconstruction
Algebraic Condition for Second Order Reconstruction.

Accuracy Condition

- Reconstruction must recover polynomials of degree one.
- This means that for all $\sigma \in \mathbb{R}^d$

$$\sigma = \sum_{\beta} s_{\alpha\beta} (h_{\alpha\beta} \cdot \sigma)$$

- This equation with unknowns $s_{\alpha\beta}$ defines a family of admissible reconstructions.
General Solution of Consistent Reconstruction

Consistency condition in matrix form

- Define geometric matrix $H_\alpha$ with rows $h_{\alpha\beta}$
- Define reconstruction matrix $S_\alpha$ with columns $s_{\alpha\beta}$
- Consistency condition becomes the linear matrix equation

$$S_\alpha H_\alpha = I_d$$

- General solution $S_\alpha = \hat{S}_\alpha + \Lambda_\alpha B_\alpha$
  - $\hat{S}_\alpha$ is a particular solution.
  - $B_\alpha$ is a maximum rank solution of $B_\alpha H_\alpha = 0$. 
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Observation of Instabilities in 3D Applications

Experience with Cedre

- Observation: Computations on unstructured grids are less accurate than those on cartesian grids.
- Hypothesis: Loss of accuracy is caused by slope limiters.
- Investigation: Perform same computations without limiters.
- Result: Computations break down on unstructured grids.

Principal Questions

- Do these instabilities exist for the linear advection equation?
- Do these instabilities exist in the semi-discrete case?
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Model Equation

- Examine the problem for the linear advection equation.

\[ \partial_t u(x, t) + c \cdot \nabla u(x, t) = 0 \]

- Spatial discretization transforms this into an ordinary differential equation.

\[ \frac{d u(t)}{d t} = J u(t) ; u = (u_1, \ldots, u_N) \]

Stability Condition

\[ ||u(t)|| \leq C \ ||u(0)|| \text{ for all times } t \geq 0 \]
Numerical Investigation of Spectra

Apply spatial MUSCL discretization to linear advection and compute spectra of the matrix $J$.

2D triangular mesh

3D tetrahedral mesh
Conclusion

**Diagnosis for Semi-discrete Linear Advection**

- Eigenvalues with positive real part in 3D on tetrahedral meshes: Spatial discretization is unstable.
- No instabilities for cartesian and deformed cartesian meshes.
- No instabilities for the first order scheme.

**Conclusion**

- Instabilities arise from the combination of linear reconstruction and unstructured meshes.
- It is necessary to examine the influence of the slope reconstruction on stability.
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A rigorous stability criterion is too difficult to obtain for the Muscl Scheme. 

Choose a consistent slope in each cell $\mathcal{T}_\alpha$ such that 

$$\sum_\alpha |\mathcal{T}_\alpha| \frac{d u^2_\alpha}{dt}$$

becomes as small as possible. $|\mathcal{T}_\alpha|$ is the volume of cell $\mathcal{T}_\alpha$. 
The time derivative of the energy can be written as a sum

$$\sum_{\alpha} |T_{\alpha}| \frac{d u_{\alpha}^2}{dt} = \sum_{\alpha=1}^{N} \{ \Theta_{\alpha}(u) + \Phi_{\alpha}(u) \}$$

$$\Theta_{\alpha}(u) = - \sum_{\beta} (c \cdot n_{\alpha\beta}^+) (u_{\beta} - u_{\alpha})^2 \leq 0$$

$$\Phi_{\alpha}(u) = 2 \sum_{\beta} \sum_{\gamma} (c \cdot n_{\alpha\beta}^+) (u_{\beta} - u_{\alpha}) k_{\alpha\beta} \cdot s_{\alpha\gamma} (u_{\gamma} - u_{\alpha})$$
Local Criterion for Reconstruction

### Transform the Problem to Obtain Local Criterion

- Define matrix $R_\alpha$ in cell $T_\alpha$ with entries $r_{\beta\gamma}^{(\alpha)} = k_{\alpha\beta} \cdot s_{\alpha\gamma}$.
- Define matrix $K_\alpha$ with rows $k_{\alpha\beta}$ then $R_\alpha = K_\alpha S_\alpha$.
- The term $\Theta_\alpha (u)$ in the time derivative is always non-positive.
- The term $\Phi_\alpha (u)$ depends linearly on $R_\alpha = K_\alpha S_\alpha$.

### Strategy to Select the Slope

- Choose $S_\alpha$ that minimizes $\| K_\alpha S_\alpha \|$ under the constraint $S_\alpha H_\alpha = I_d$. 
Local Reconstruction
Graphical Representation of the Local Reconstruction

- $R_\alpha = K_\alpha S_\alpha$ maps $u_\beta - u_\alpha$ to $u_{\alpha\gamma} - u_\alpha$ where
  \[ u_{\alpha\gamma} = u_\alpha + k_{\alpha\gamma} \cdot \sigma_\alpha \]

- If $R_\alpha = 0$, then
  \[ \sum_\alpha |\mathcal{T}_\alpha| \frac{du_\alpha^2}{dt} \leq 0 \]

\[ \rightarrow \text{stability of the first order upwind scheme!} \]
Minimization Property of the Least Squares Reconstruction

Minimization problem

- Find consistent reconstruction that minimizes $\|K_{\alpha}S_{\alpha}\|:

$$\min \{\|K_{\alpha}S_{\alpha}\| ; S_{\alpha}H_{\alpha} = I_d\}$$

Solution exists for a large family of norms

- Least-squares slope solves $\min \{\|K_{\alpha}S_{\alpha}\| ; S_{\alpha}H_{\alpha} = I_d\}$ for all unitarily invariant norms.
- This includes Spectral Norm, Frobenius Norm, Trace Norm.
- Least-squares reconstruction is optimal for the criterion.
Outline of the Proof I

- General form of reconstruction
  \[ S_\alpha = \hat{S}_\alpha + \Lambda_\alpha B_\alpha. \]

- Least Squares Reconstruction
  \[ \hat{S}_\alpha = (H_\alpha^t H_\alpha)^{-1} H_\alpha^t. \]

- The matrix \( \hat{S}_\alpha \) satisfies
  \[ B_\alpha \hat{S}_\alpha^t = 0, \hat{S}_\alpha B_\alpha^t = 0. \]

- This implies
  \[ K_\alpha S_\alpha S_\alpha^t K_\alpha^t = K_\alpha \hat{S}_\alpha \hat{S}_\alpha^t K_\alpha^t + K_\alpha \Lambda_\alpha B_\alpha B_\alpha^t \Lambda_\alpha^t K_\alpha^t. \]
Outline of the proof II

**Definition**

A matrix norm $\| \cdot \|$ is called *unitarily invariant* if $\| A \| = \| UAV \|$ for any matrix $A$ and arbitrary unitary matrices $U$ and $V$.

**Definition**

A norm $g$ on $\mathbb{R}^k$ is called a *symmetric gauge function* if it is an absolute and permutation invariant vector norm:

$g(|x_1|, \ldots, |x_k|) = g(x_1, \ldots, x_k)$ and $g(x) = g(Px)$ for any permutation matrix $P$. 
Theorem

Any unitarily invariant matrix norm $||A||$ can be written as a symmetric gauge function of the vector of singular values of $A$

$$||A|| = g(\varsigma_1(A), \ldots, \varsigma_k(A)) .$$

Theorem

Let $\varsigma_1 (A) \geq \cdots \geq \varsigma_k (A)$ be the singular values of $A$ and $\lambda_1 (AA^*) \geq \cdots \geq \lambda_k (AA^*)$ the eigenvalues of $AA^*$ (They are the same as those of $A^*A$). Then

$$\varsigma_i (A) = \sqrt{\lambda_i (AA^*)} , \ 1 \leq i \leq k .$$
Outline of the Proof IV

**Theorem (Weyl)**

Let $P$ and $Q$ be hermitian matrices. If $Q$ is positive semi-definite, then

$$
\lambda_i (P) \leq \lambda_i (P + Q) \quad , 1 \leq i \leq k.
$$

Weyl’s Theorem shows

$$
\lambda_i \left( K_\alpha \hat{S}_\alpha \hat{S}_\alpha^t K_\alpha^t \right) \leq \lambda_i \left( K_\alpha \hat{S}_\alpha \hat{S}_\alpha^t K_\alpha^t + K_\alpha \Lambda_\alpha B_\alpha B_\alpha^t \Lambda_\alpha K_\alpha^t \right)
$$

Therefore the singular values of $K_\alpha S_\alpha$ satisfy

$$
\varsigma_i \left( K_\alpha \hat{S}_\alpha \right) \leq \varsigma_i \left( K_\alpha S_\alpha \right) \quad , 1 \leq i \leq k.
$$
Outline of the proof V

**Theorem**

Any symmetric gauge function $g$ is a monotone vector norm.

Let $x, y \in \mathbb{R}^k$.

$$|x_i| \leq |y_i|, \ 1 \leq i \leq k, \text{ implies } g(x_1, \ldots, x_k) \leq g(y_1, \ldots, y_k).$$

The combination of the theorems gives

**Theorem (Minimization property)**

Let $\| . \|$ be any unitarily invariant matrix norm and $S_\alpha$ any reconstruction matrix $S_\alpha$ that is consistent, i.e. that is a solution of $S_\alpha H_\alpha = I_d$. Then the least squares reconstruction matrix $\hat{S}_\alpha$ satisfies

$$\|K_\alpha \hat{S}_\alpha\| \leq \|K_\alpha S_\alpha\|$$
Influence of the Stencil Size on Stability
A qualitative result for the least squares reconstruction

Let $\hat{S}_\alpha$ be the matrix of the least-squares reconstruction.

What happens to the unitarily invariant matrix norms of $\|K_\alpha \hat{S}_\alpha\|$ if points are added to the stencil?

**Theorem**

- New reconstruction matrix $\tilde{S}_\alpha$ satisfies $\|K_\alpha \tilde{S}_\alpha\| \leq \|K_\alpha \hat{S}_\alpha\|$.
- and even $\|K_\alpha \tilde{S}_\alpha\| < \|K_\alpha \hat{S}_\alpha\|$ under certain conditions.
- Conclusion: Larger stencils lead to more robust schemes.
Outline of the Proof I

- In cell $T_\alpha$, let the matrix $H_\alpha$ have rows $\{h_{\alpha\beta_1}, \ldots, h_{\alpha\beta_n}\}$.
- Add $l$ new cells $\{T_{\gamma_1}, \ldots, T_{\gamma_l}\}$ to the reconstruction stencil.
- Define matrix $\tilde{H}_\alpha$ with rows $\{h_{\alpha\gamma_1}, \ldots, h_{\alpha\gamma_l}\}$.
- New geometric matrix is given by
  \[ \tilde{H}_\alpha^t = [H_\alpha^t | \tilde{H}_\alpha^t]. \]

- The matrix $\tilde{H}_\alpha$ satisfies
  \[ \left(\tilde{H}_\alpha^t \tilde{H}_\alpha\right)^{-1} = \left(H_\alpha^t H_\alpha + \tilde{H}_\alpha^t \tilde{H}_\alpha\right)^{-1}. \]
Outline of the proof II

- Least squares reconstruction matrices

\[ \tilde{S}_\alpha = \left( \tilde{H}_\alpha^t \tilde{H}_\alpha \right)^{-1} \tilde{H}_\alpha^t, \quad \hat{S}_\alpha = \left( H_\alpha^t H_\alpha \right)^{-1} H_\alpha^t. \]

- The matrix \( \tilde{S}_\alpha \) satisfies

\[ K_\alpha \tilde{S}_\alpha \tilde{S}_\alpha^t K_\alpha^t = K_\alpha \left( \tilde{H}_\alpha^t \tilde{H}_\alpha \right)^{-1} K_\alpha^t. \]

- The Sherman-Morrison-Woodbury identity shows

\[ K_\alpha \left( H_\alpha^t H_\alpha \right)^{-1} K_\alpha^t = K_\alpha \left( H_\alpha^t H_\alpha + \tilde{H}_\alpha^t \tilde{H}_\alpha \right)^{-1} K_\alpha^t + K_\alpha \left( H_\alpha^t H_\alpha \right)^{-1} \tilde{H}_\alpha^t \left( I + \tilde{H}_\alpha \left( H_\alpha^t H_\alpha \right)^{-1} \tilde{H}_\alpha^t \right)^{-1} \tilde{H}_\alpha \left( H_\alpha^t H_\alpha \right)^{-1} K_\alpha^t. \]
Outline of the Proof III

Weyl’s Theorem shows

\[ \lambda_i \left( K_\alpha \tilde{S}_\alpha \tilde{S}_\alpha^t K_\alpha^t \right) \leq \lambda_i \left( K_\alpha \hat{S}_\alpha \hat{S}_\alpha^t K_\alpha^t \right) \]

Therefore the singular values of \( K_\alpha \tilde{S}_\alpha \) satisfy

\[ \varsigma_i \left( K_\alpha \tilde{S}_\alpha \right) \leq \varsigma_i \left( K_\alpha \hat{S}_\alpha \right), \quad 1 \leq i \leq k. \]

This implies for any unitarily invariant matrix norm

\[ \left\| K_\alpha \tilde{S}_\alpha \right\| \leq \left\| K_\alpha \hat{S}_\alpha \right\|. \]
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Practical Conclusions

- Theoretical results suggest (but no complete proof yet!)
  - It is not reasonable to expect any consistent reconstruction to give better stability than the Least Squares Method.
  - Use of larger stencils than first neighborhood is recommended in three dimensions.

Numerical Study for Linear Advection

- There are grids where the Least Squares Method is stable but alternative methods are not.
- In the case of piecewise linear reconstruction the second neighborhood turns out to be sufficient for stability on tetrahedral meshes.
Practical Implementation

**Practical Solution for Solver CEDRE**

- CEDRE handles large grids by parallel computing and grid partitioning.
- Large reconstruction stencils are not easy to implement.

**Practical Solution for Consistent and Stable Reconstruction**

- Compute least-squares slope on first neighborhood.
- Take weighted average of slopes over first neighborhood.
- Numerical computations of spectra show that this method is stable.
- This method is consistent.
Application to Compressible Gas Dynamics
Three-dimensional Flow Over a Deep Cavity.

With the new scheme, the simulation of a three-dimensional subsonic flow over a deep cavity is possible without slope limiters on tetrahedral grids.

Entropy Distribution

Pressure Signal in the Cavity
Application to Compressible Gas Dynamics
Three-dimensional Flow Over a Deep Cavity.

Comparison of the spectra of a pressure signal in the upstream wall of the cavity. The limiter used is the best limiter available in CEDRE: expensive!!

Result without limiters

Result with limiters
Application to Compressible Gas Dynamics
Three-dimensional Jet Noise Computations

Influence of slope limiters on a jet noise computation on tetrahedral grids.

- Blue line: new scheme - limiters are active only inside the nozzle.
- Red and green line: new and old scheme with slope limiters active on the entire domain.

Sensor Positions

Pressure Signal at Sensor 15
Summary

- Without limiters, spatial MUSCL discretization of linear advection leads to instabilities on unstructured three-dimensional grids containing tetrahedra and prisms.
- Instabilities can be eliminated by the least-squares method and larger stencils than the first neighborhood.

Outlook

- The criterion can be used for higher order reconstructions.
For Further Reading I

F. Haider, J.P. Croisille, B. Courbet
Stability Analysis of the Cell Centered Finite-Volume Muscl Method on Unstructured Grids.
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